

Quasiparticle transport in the vortex state of d-wave superconductors.

I. Vekhter¹, J. P. Carbotte², E. J. Nicol¹

¹*Department of Physics, University of Guelph, Guelph, Ontario N1G 2W1, Canada*

²*Department of Physics and Astronomy, McMaster University, Hamilton, Ontario L8S 4M1, Canada*
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We calculate the magnetic field and temperature dependence of the Raman response, superfluid density and the NMR relaxation rate in the vortex state of a d-wave superconductor arising from the Doppler energy shift of extended quasiparticle states. Our results are valid both at low temperatures, where we observe scaling with variable $TH^{-1/2}$ and obtain explicit form of the scaling functions, and beyond this region. We derive a universal frequency dependent scaling relation for the Raman response, and discuss the breakdown of the single relaxation rate approach to NMR response.

74.22.-h, 74.60.-w, 74.25.Gz, 74.24.Nf

In the last few years there has emerged a consensus regarding the d-wave symmetry of the order parameter in the hole doped high- T_c cuprates. While in conventional, s-wave, superconductors a finite energy gap exists everywhere on the Fermi surface, the d-wave order parameter has lines of nodes, which leads to a gapless excitation spectrum along certain directions in momentum space. Consequently the low temperature behavior of the thermodynamic and transport coefficients in the high- T_c materials are qualitatively different from those of the s-wave compounds. Properties of the vortex state in “unconventional” superconductors also differ significantly from those obtained within the framework of s-wave superconductivity, and a clear understanding of these properties is essential for interpretation of the experimental results and a better grasp of the novel physics associated with the superconducting state of the high- T_c materials.

Volovik [1,2] pointed out that while in an s-wave superconductor the density of states is determined at low fields by the quasiparticles in the vortex core, both the density of states and the entropy in d-wave superconductors are dominated by the extended quasiparticle states which exist even at zero temperature in the nodal directions of the order parameter. A remarkable consequence of this behavior is that the specific heat of such a superconductor varies as \sqrt{H} rather than linearly in the applied field. Other authors [3] used the Dirac form of the low-energy excitation spectrum of nodal quasiparticles to demonstrate that thermal and transport coefficients exhibit scaling with $TH^{-1/2}$. However, strictly speaking, the analysis applies only to clean superconductors, and the energy spectrum is only Dirac-like at energies small compared to the gap amplitude Δ_0 ; a crossover to a different scaling regime followed by the breakdown of scaling have been predicted at $(T/T_c)(H_{c2}/H)^{1/2} \sim 1$ [2,3].

Very recently Kübert and Hirschfeld [4,5] placed these scaling arguments in the framework of the Green’s function formalism capable of treating both the energies of order of the gap and the effects of disorder. These authors argued that for a short coherence length superconductor the typical spacing of the energy levels in the core, Δ_0^2/E_f , where E_f is the Fermi energy, is large so that only one or a few states (if any) exist there, and suggested that the contribution of the vortex cores to the transport coefficients is negligible over a wide region of H and T . They proposed to account for the effect of the magnetic field on the extended states semiclassically by introducing a Doppler shift due to circulating supercurrents, which for $H \ll H_{c2}$ are approximated by the superfluid velocity field around a single vortex $\mathbf{v}_s = \hbar\hat{\theta}/2mr$, where r is the distance from the center of the vortex and θ is the azimuthal angle in real space. The authors of Ref. [5] investigated in detail the breakdown of scaling of both specific heat and the thermal conductivity with increased impurity scattering, and the results agreed remarkably well with recent experiments [6].

In this work we use the same approach to examine the effect of magnetic fields $H_{c1} \leq H \ll H_{c2}$ on the Raman response, superfluid density, Knight shift, and NMR relaxation rate in a d-wave superconductor. These are issues of considerable experimental interest: recently Blumberg *et al.* [7] analyzed the changes induced by a magnetic field in the electronic part of the Raman response; NMR relaxation rates and the Knight shift are measured in fields of up to 10T [8], in the vortex state, while muon spin rotation (μ SR) is used to determine the low temperature penetration depth in a magnetic field [9], which is related to the superfluid density. In our analysis we obtain scaling similar to that suggested in Ref. [3,2] and give the explicit form of the scaling functions at low temperatures, however our results remain valid in a wider parameter range. We do not include impurity scattering in the present calculations, if included, it affects only the extreme low frequency (low temperature) part of the Raman (NMR) response, and the results presented here remain valid beyond this narrow region.

We employ the single particle Green’s function which is obtained by introducing the Doppler shift into the BCS function [10,4]

$$G(\mathbf{k}, \omega_n; \mathbf{r}) = -\frac{(i\omega_n - \mathbf{v}_s \cdot \mathbf{k})\tau_0 + \Delta_{\mathbf{k}}\tau_1 + \zeta_{\mathbf{k}}\tau_3}{\omega_n^2 + \zeta_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}, \quad (1)$$

where ω_n is the fermionic Matsubara frequency, $\zeta_{\mathbf{k}}$ is the energy of a quasiparticle with momentum \mathbf{k} , measured with respect to the Fermi level, and τ_i are Pauli matrices. The Green's function depends on the coordinate \mathbf{r} in real space via the superfluid velocity \mathbf{v}_s . Now thermal and transport coefficients can be calculated using the standard approach [11], however, they become local quantities which have to be averaged over a unit cell of the vortex lattice. Following Ref. [4] we approximate this unit cell by a circle of radius R , where $2R = \xi_0 \sqrt{2\pi} a^{-1} (H_{c2}/H)^{1/2}$ is the intervortex spacing, and a is a geometric constant of order unity, so that the average of a quantity $f(\mathbf{r})$ is given by

$$f(H) = \frac{1}{\pi R^2} \int d^2 r f(r, \theta). \quad (2)$$

Here we consider a model cylindrical Fermi surface and an experimental arrangement with the magnetic field parallel to the axis of the cylinder, $\mathbf{H} \parallel c$.

Raman response. The Raman intensity is proportional to the imaginary part of the zero momentum density-density correlation function

$$\chi(i\Omega_n) = -\gamma_R^2 T \sum_{\mathbf{k}, \omega_m} f_s^2(\hat{\mathbf{k}}) \text{Tr} [\tau_3 G(\mathbf{k}, i\omega_m) \tau_3 G(\mathbf{k}, i\omega_m - i\Omega_n)], \quad (3)$$

where $\gamma_R f_s(\hat{\mathbf{k}})$ is the Raman vertex. In Eq.(3) vertex corrections due to Coulomb screening have been ignored, they appear only in the fully symmetric channel (A_{1g}), while we are interested primarily in the B_{1g} and B_{2g} scattering geometries. After doing the Matsubara sum and analytically continuing the response function to real frequencies, we obtain the local Raman response

$$\chi''(\Omega; \mathbf{r}) = \frac{1}{2} \gamma_R^2 N(0) \int_{FS} d\hat{\mathbf{k}} f_s^2(\hat{\mathbf{k}}_f) \frac{1}{\Omega \sqrt{(\Omega/2)^2 - \Delta_{\mathbf{k}}^2}} \left[\tanh \frac{\Omega - 2\mathbf{v}_s \cdot \hat{\mathbf{k}}_f}{4T} + \tanh \frac{\Omega + 2\mathbf{v}_s \cdot \hat{\mathbf{k}}_f}{4T} \right], \quad (4)$$

where the integration is over the Fermi surface, and k_f is the Fermi momentum. Note that the kernel of the integral is identical to that in zero field, and that the field dependence is only in the thermal factors. Indeed, as the Doppler shift is the same for the quasiparticles absorbing and emitting photons the difference in energy between the initial and the final states remains unchanged, whereas the thermal factors, which depend on the local value of the chemical potential, are strongly affected by the magnetic field.

To obtain the field dependent Raman intensity we spatially average the local response given in Eq.(4) according to Eq.(2). A crucial observation is that since the spatial average is performed over all directions of the superfluid velocity, the result does not depend on specific position $\hat{\mathbf{k}}_f$ at the Fermi surface. Consequently, spatial averaging decouples from the integration over the Fermi surface, and we arrive at the following universal scaling relation for the Raman intensity

$$\chi''(\Omega; H) = \chi''(\Omega; 0) F(\Omega/2T, E_H/T)/F(\Omega/2T, 0), \quad (5)$$

where $E_H = a\Delta_0 \sqrt{H/H_{c2}}$ is the typical quasiparticle Doppler shift, which is the energy scale introduced by the magnetic field, and

$$F(x, y) = \frac{1}{\pi} \int_0^1 z dz \int_0^{2\pi} d\theta \tanh\left(\frac{x}{2} - \sqrt{\frac{\pi}{8}} \frac{y}{z} \cos\theta\right) \quad (6)$$

is a generalization of the thermal function $F(\epsilon/T, 0) = \tanh(\epsilon/2T)$. In general the function F has to be evaluated numerically, however it can be obtained analytically in the important limit $T = 0$ when it depends only on the ratio x/y :

$$F_0(w) = \begin{cases} 1 - 1/(2w^2) & w \geq 1; \\ \pi^{-1} [(2 - w^{-2}) \arcsin w + \sqrt{w^{-2} - 1}], & w \leq 1, \end{cases} \quad (7)$$

here $w = (2/\pi)^{1/2}(x/y)$. Then the ratio of Raman intensities in *any* channel at low temperatures is given by

$$\frac{\chi''|_{T=0}(\Omega; H)}{\chi''|_{T=0}(\Omega; 0)} = F_0\left(\frac{\Omega}{\sqrt{2\pi} E_H}\right). \quad (8)$$

The intensity is most strongly affected for the Raman shifts below the average Doppler shift E_H , where the scaling function F_0 is almost linear in $\Omega H^{-1/2}$, while for large frequencies the field dependent correction is small and linear in the applied field. In Fig.1 we also show that the main features of the scaling function remain robust at low temperatures.

So far we have made no assumptions regarding the specific symmetry of the order parameter or the particular Raman geometry. Assuming a $d_{x^2-y^2}$ -wave symmetry, we obtain that the intensity at small frequencies $\Omega \ll E_H$ in the B_{1g} and B_{2g} channels become quartic and quadratic, respectively, compared to cubic and linear dependence in the absence of field. It also follows that the height of the $2\Delta_0$ peak in the B_{2g} channel decreases linearly with the magnetic field. Finally the integrated normalized Raman intensity $\int_0^\infty d\Omega [\Delta\chi''(\Omega; H)/\chi''(\Omega; 0)]$ scales with \sqrt{H} , while the integral of the change in the signal itself depends on the particular Raman geometry [12].

Superfluid density. Using the Green's function given in Eq.(1) we obtain that the relative change in superfluid density is given by the generalized Yosida function [13]

$$\frac{\delta n_s(T, H)}{n_s(0, 0)} = \frac{1}{2N(0)} \sum_{\mathbf{k}} \frac{\partial}{\partial E_{\mathbf{k}}} F\left(\frac{E_{\mathbf{k}}}{T}, \frac{E_H}{T}\right), \quad (9)$$

where $E_{\mathbf{k}} = \sqrt{\zeta_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}$, and the function F was defined in Eq.(6). It is clear from the zero-temperature limit F_0 of the thermal function F , and from Fig.1, that at low temperatures $T \ll E_H$ the energy scale determining the range of $F(E_{\mathbf{k}}/T, E_H/T)$ as a function of $E_{\mathbf{k}}$ is the magnetic energy E_H rather than T , so that the behavior of transport coefficients is drastically different from the $H = 0$ case. As we show in Fig.2, while in absence of magnetic field the superfluid density n_s decreases linearly with T due to the linear low-energy density of states of a superconductor with lines of nodes of the gap, the low temperature behavior in the applied field becomes

$$\frac{\delta n_s(T, H)}{n_s(0, 0)} \approx \sqrt{\frac{8}{\pi} \frac{E_H}{\Delta_0}} + \frac{2}{9} \frac{\sqrt{2\pi} T^2}{E_H \Delta_0}, \quad (10)$$

and the temperature dependent term shows scaling with $T^2 H^{-1/2}$. In all the numerical work we have used the BCS value $\Delta_0 = 2.14T_c$. The superfluid density can be extracted from either optical conductivity in the magnetic field, which has not been measured yet, or from the μSR experiments, which determine the London penetration depth $\lambda(H)$ as $T \rightarrow 0$. The exact relationship between the penetration depth and the superfluid density in the mixed state is not yet clearly understood, since nonlinear [14] and nonlocal [15] effects are believed to be important, however, the non-linear, in H , behaviour similar to that given in Eq.(10) has been obtained numerically when both of these effects are included [16]. The inset of Fig.2 shows the data of Ref. [9] for a three-crystal mosaic of $YBCO$ with our best linear, in \sqrt{H} , fit, which gives $\lambda(0) = 1142\text{\AA}$ and $H_{c2}/a^2 = 88T$, compared to $\lambda(0) = 1155\text{\AA}$ used in Ref. [9] for a linear in H fit. Because of nonlinear effects the obtained value for H_{c2} is a low-end estimate for the upper critical field.

The scaling behavior given by Eq.(10), is shown in Fig.3 where the numerical results follow the scaling curve up to $T/E_H \approx (T/\Delta_0)(H_{c2}/H)^{1/2} \sim 1$. For larger T/E_H results obtained for small E_H and, consequently, low temperatures still scale, while for larger fields the scaling is broken since corresponding temperatures become high.

We note that in conventional linear response theory the relative change to the spin part of the Knight shift in fields $\mu_B H \ll T$ is also given by Eq.(9). However, in the present problem there is an additional magnetization due to the vortex lattice, which has to be computed from the grand potential [17]. This contribution is small in the temperature dominated regime $T \gg E_H$, while at low temperatures $M \approx (a^2 \pi^2 T^2 \Delta_0 N(0))/(6E_H H_{c2})$.

Spin-lattice relaxation rate. Short-range antiferromagnetic correlations, which are believed to be important for the Cu NMR relaxation rate in the cuprates, cancel on the oxygen sites, resulting in a normal Korringa behavior seen in experiment [8,18]. Then the relaxation rate T_1^{-1} can be calculated using the low frequency limit of the uniform spin susceptibility, yielding

$$\frac{1}{T_1(r, \theta)T} = \frac{1}{2T_1^{(c)}T_c} \int_{-\infty}^{+\infty} N^2(E) \frac{\partial}{\partial E} \tanh \frac{E - \mathbf{v}_s \cdot \mathbf{k}}{2T}, \quad (11)$$

where $T_1^{(c)}$ is the relaxation time at T_c , and $N(E)$ is the superconducting density of states. It is important to note that the NMR experiments measure the decay of magnetization $\mathbf{M}(t) \propto \exp(-t/T_1)$, rather than T_1 directly, so that it is $\mathbf{M}(t)$ that has to be spatially averaged. For a distribution of $T_1(\mathbf{r})$ the average magnetization cannot be described by a single relaxation rate, and, if such a fit is made, the obtained value of T_1 is different depending on whether short or long time scale behavior is analyzed. At low temperatures $T \ll T_c$, if $T \geq E_H$, the times over which measurements are

done, $t \sim T_1$, correspond to short time scale in the field-dependent term, and the magnetization decay approximately follows a single relaxation rate behavior with

$$\frac{T_c T_1^{(c)}}{T_1 T} = \frac{\pi^2}{3} \frac{T^2}{\Delta_0^2} + \frac{\pi}{2} \frac{E_H^2}{\Delta_0^2} \ln \left[\frac{T}{E_H} \right]. \quad (12)$$

On the other hand, for $T_c \gg E_H \gg T$, the single relaxation time picture breaks down completely due to strong spatial variations of T_1 . Even though $\mathbf{M}(t)$ is still described by the approximate relaxation rate given by Eq.(12) for $t \ll T_1$, in the experimentally relevant region $\mathbf{M}(t) \propto \exp(-t/T_1^{(H=0)})/t^{1/2}$. We are aware that an independent analysis of NMR magnetization data, including a wider range of temperatures and fields, is being carried out, using an approach similar in spirit to this one [19].

To conclude we have presented an approach to the calculation of thermal and transport properties of d-wave superconductors in the mixed state over a wide range of temperatures and fields, considering the contribution of the extended quasiparticle states. We obtained the explicit form of the scaling functions in the low temperature regime $T \leq E_H$, and observed the breakdown of scaling at higher temperatures. Our results agree qualitatively with the μSR measurements of the penetration depth in the vortex state.

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- [1] G. E. Volovik, JETP Lett. **58**, 469 (1993) [Pis'ma Zh. Eksp, Teor. Fiz. **58**, 457 (1993)]; JETP Lett. **65**, 491 (1997) [Pis'ma Zh. Eksp, Teor. Fiz. **65**, 465 (1997)].
- [2] N. B. Kopnin and G. E. Volovik, JETP Lett. **64**, 690 (1996) [Pis'ma Zh. Eksp, Teor. Fiz. **64**, 641 (1996)].
- [3] S. H. Simon and P. A. Lee, Phys. Rev. Lett. **78**, 1548 (1997); *ibid*. **78**, 5029 (1997); G. E. Volovik and N. B. Kopnin, *ibid* **78**, 5028 (1997).
- [4] C. Kübert and P. J. Hirschfeld, Sol. St. Comm. **105**, 459 (1998).
- [5] C. Kübert and P. J. Hirschfeld, cond-mat/9802274; cond-mat/9801105, to be published in Phys. Rev. Lett.
- [6] M. Chiao *et al.*, unpublished.
- [7] G. Blumberg, M. Kang, and M. V. Klein, Phys. Rev. Lett. **78**, 2461 (1997); G. Blumberg, private communication.
- [8] M. Takigawa *et al.*, Phys. Rev. **B 43**, 247 (1991).
- [9] J. E. Sonier *et al.*, Phys. Rev. **B 55**, 11789 (1997); J. E. Sonier *et al.*, Phys. Rev. Lett. **79**, 2875 (1997).
- [10] K. Maki and T. Tsuneto, Progr. Theor. Phys. **27**, 228 (1960).
- [11] See, for example, A. A. Abrikosov, L. P. Gorkov, I. E. Dzyaloshinskii, *Methods of Quantum Field Theory in Statistical Physics*, (Dover, New York, 1963).
- [12] We are grateful to G. Blumberg for suggestion to address this issue.
- [13] K. Yosida, Phys. Rev. **110**, 769 (1958).
- [14] S. K. Yip and J. A. Sauls, Phys. Rev. Lett. **69**, 2264 (1992).
- [15] I. Kosztin and A. Leggett, Phys. Rev. Lett. **79**, 135 (1997).
- [16] M. H. S. Amin, I. Affleck and M. Franz, cond-mat/9712218.
- [17] H. Won and K. Maki, Europhys. Lett. **30**, 421 (1995).
- [18] P. C. Hammel *et al.* Phys. Rev. Lett. **63**, 1992 (1989).
- [19] R. Wortis, Ph. D. Thesis, University of Illinois at Urbana-Champaign, 1998; R. Wortis *et al.*, unpublished.

FIG. 1. Zero temperature scaling function $F_0(w)$ (solid line). The result of numerical evaluation of the function F , given in Eq.(6), is plotted for $T = 0.03\Delta_0$ for comparison (dashed line).

FIG. 2. Superfluid density n_s as a function of reduced temperature for different magnetic fields: $E_H = 0$ (solid line), $E_H = 0.1T_c$ (dashed line), $0.2T_c$ (long-dashed line), $0.3T_c$ (dot-dashed line). Inset: zero-temperature penetration depth data from μSR experiment plotted vs. \sqrt{H} . Solid line: best linear fit. Slope corresponds to $H_{c2}/a^2 = 88T$.

FIG. 3. Full numerical evaluation of the superfluid density from Eq.(9). Dashed line: low temperature result from Eq.(10).





